Part 4. Integration

Definition

Definition 4.1 *A **partition** $\mathcal{P}$ of $[a,b]$ is a finite set of points $\{x_0, x_1, ..., x_n\}$ with $a = x_0 < x_1 < ... < x_n = b$.*

Example 4.2 $\left\{0, \frac{1}{4}, \frac{1}{3}, \frac{7}{8}, 1\right\}$ *and* $\left\{0, \frac{1}{100}, \frac{3}{100}, 1\right\}$ *are partitions of* $[0,1]$.

The following definitions are not universal but I find them very useful.

Definition 4.3 *For each $n \geq 1$ the **arithmetic partition** of $[a,b]$ is*

$$\mathcal{P}_n = \left\{a + \frac{(b-a)}{n} i : 0 \leq i \leq n\right\}.$$

Example 4.4 *For the interval $[2,3]$ with $n = 4$, the arithmetic partition is*

$$\mathcal{P}_4 = \left\{2, \frac{9}{4}, \frac{10}{4}, \frac{11}{4}, 3\right\} = \left\{2 + \frac{3-2}{4} i : 0 \leq i \leq 4\right\}.$$

Definition 4.5 *Assume $0 < a < b$. For each $n \geq 1$ the **geometric partition** of $[a,b]$ is*

$$\mathcal{Q}_n = \left\{a \left(\frac{b}{a}\right)^{i/n} : 0 \leq i \leq n\right\} = \left\{a\eta^i : 0 \leq i \leq n\right\},$$

where $\eta = (b/a)^{1/n}$.

Example 4.6 *For the interval $[2,10]$ with $n = 3$, the geometric partition is*

$$\mathcal{Q}_3 = \left\{2, 2 \times 5^{1/3}, 2 \times 5^{2/3}, 10\right\} = \left\{2 \left(\frac{10}{2}\right)^{i/3} : 0 \leq i \leq 3\right\}.$$

A partition of $[a,b]$ divides the interval into n sub-intervals $[x_{i-1}, x_i]$ for $1 \leq i \leq n$. In an arithmetic partition

$$x_i = x_{i-1} + \frac{b-a}{n}$$

for all $1 \leq i \leq n$. In a geometric partition $x_i = \eta x_{i-1}$ for all $1 \leq i \leq n$.

Let f be a **bounded** function on $[a,b]$, so there exist m and M such that $m \leq f(x) \leq M$ for all $x \in [a,b]$. For each $1 \leq i \leq n$ consider the set

$$\{f(x) : x \in [x_{i-1}, x_i]\}.$$

It is a non-empty set, bounded above by M, and from below by m. Hence by the Completeness of $\mathbb{R}$ the set has a least upper bound and greatest lower bound. Therefore we can make the definition

Definition 4.7 *For* $1 \leq i \leq n$ *set*

$$M_i = \text{lub}\,\{f(x) : x \in [x_{i-1}, x_i]\}\,,$$

and

$$m_i = \text{glb}\,\{f(x) : x \in [x_{i-1}, x_i]\}\,.$$

Definition 4.8 *For* f *bounded on* $[a,b]$ *the* ***Upper Sum*** *for* f *with the partition* $\mathcal{P}$ *is*

$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i\,(x_i - x_{i-1})$$

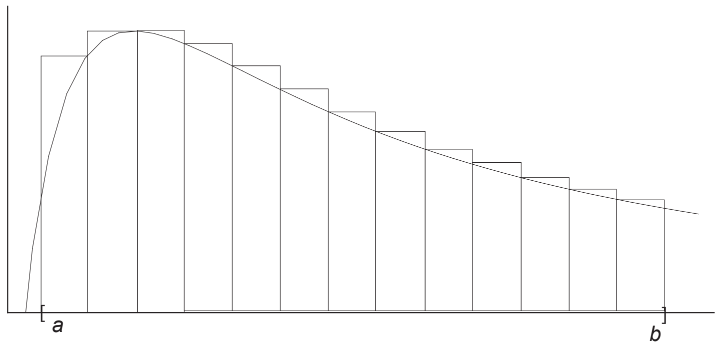
and the ***Lower Sum*** *is*

$$L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i\,(x_i - x_{i-1})\,.$$

Advice for the exam. I'm sure that it need not be stressed that you need to remember these definitions for the exam, but how?

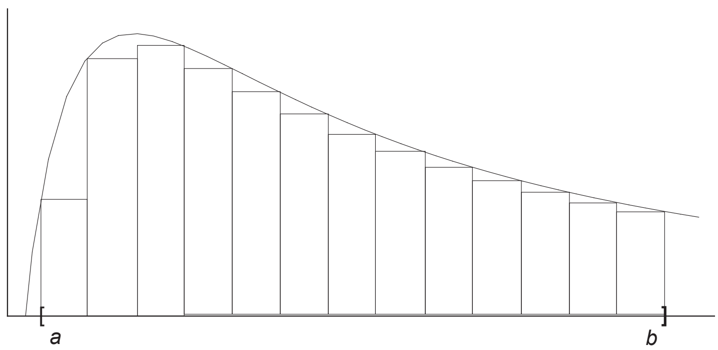
What do the Upper and Lower sums represent? Both are summations of terms of the form $m\,(s-t)$ which you might think of as the *area* of a rectangle with base the interval $[t, s]$ and height m. Thus the Upper and Lower sums represent the *sums of areas of rectangles* with bases $[x_{i-1}, x_i]$. These bases can only intersect at the end points so the Upper and Lower sums represent the areas of unions of disjoint rectangles. How are these areas related to the function f?

Example 4.9 *An upper sum represents the sum of the areas of the rectangles in the diagram.*



So you can think of the upper sum as ***over***-estimating the area under the graph.

Example 4.10 *A lower sum represents the sum of the areas of the rectangles in the diagram.*



So you can think of the lower sum as ***under***-estimating the area under the graph.

Be careful, I have drawn a “nice, smooth” function, the situation might look different with more “pathological” functions. Also be careful because I have drawn only non-negative functions. I suggest you look at what happens if the function should be negative for some x. Some of the rectangles may well have a *negative area!*

If we could assign a measure of size to the region under a graph we might expect it to be

- ***less*** than the areas measured by $U(\mathcal{P}, f)$ for all $\mathcal{P}$, yet
- ***greater*** than those measured by $L(\mathcal{P}, f)$ for all $\mathcal{P}$.

Thus we will be interested in the

- "smallest values" taken by $U(\mathcal{P}, f)$ as $\mathcal{P}$ varies, and the
- "largest values" of $L(\mathcal{P}, f)$ as $\mathcal{P}$ varies.

Examples

Example 4.11 *Let $f : [2,4] \to \mathbb{R} : x \mapsto 3x - 8$. Find $U(\mathcal{P}_n, f)$ and $L(\mathcal{P}_n, f)$, for the arithmetic partitions $\mathcal{P}_n$.*

Solution In the notation above, $a = 2$, $b = 4$, so $b - a = 2$ and thus

$$\mathcal{P}_n = \left\{ 2 + \frac{2i}{n} : 0 \leq i \leq n \right\}.$$

Then

$$[x_{i-1}, x_i] = \left[2 + \frac{2}{n}(i-1), 2 + \frac{2i}{n} \right] \quad \text{and} \quad x_i - x_{i-1} = \frac{2}{n}$$

for all $i \geq 1$. On the interval $[2,4]$ the function $f(x) = 3x - 8$ is increasing so

$$\begin{aligned} M_i &= \operatorname*{lub}_{[x_{i-1}, x_i]} f(x) = f(x_i) = 3x_i - 8 \\ &= 3\left(2 + \frac{2i}{n}\right) - 8 = \frac{6i}{n} - 2. \end{aligned}$$

Similarly

$$\begin{aligned} m_i &= \operatorname*{glb}_{[x_{i-1}, x_i]} f(x) = f(x_{i-1}) = 3x_{i-1} - 8 \\ &= 3\left(2 + \frac{2(i-1)}{n}\right) - 8 = \frac{6(i-1)}{n} - 2. \end{aligned}$$

Because M_i 'looks simpler' than m_i I first consider the upper sum:

$$\begin{aligned} U(\mathcal{P}_n, f) &= \sum_{i=1}^{n} \left(\frac{6i}{n} - 2 \right) \frac{2}{n} \\ &= \frac{12}{n^2} \sum_{i=1}^{n} i - \frac{4}{n} \sum_{i=1}^{n} 1 = \frac{12}{n^2} \frac{n(n+1)}{2} - \frac{4}{n} n \\ &= \frac{6(n+1)}{n} - 4 = 2 + \frac{6}{n}. \end{aligned}$$

Similarly,

$$L(\mathcal{P}_n, f) = \sum_{i=1}^{n} \left(\frac{6\,(i-1)}{n} - 2 \right) \frac{2}{n}.$$

You could evaluate this as we did for $U(\mathcal{P}_n, f)$ but why do the same work twice? I suggest trying to reuse work done for $U(\mathcal{P}_n, f)$ by first changing the variable to $j = i - 1$ when

$$L(\mathcal{P}_n, f) = \sum_{j=0}^{n-1} \left(\frac{6j}{n} - 2 \right) \frac{2}{n}.$$

Compared to $U(\mathcal{P}_n, f)$ this has an additional $j = 0$ term and is missing the $j = n$ term. We rearrange with the intent of rewriting the sum in terms of $U(\mathcal{P}_n, f)$:

$$\begin{aligned} L(\mathcal{P}_n, f) &= \sum_{j=1}^{n} \left(\frac{6j}{n} - 2 \right) \frac{2}{n} + \left(\frac{6 \times 0}{n} - 2 \right) \frac{2}{n} - \left(\frac{6n}{n} - 2 \right) \frac{2}{n} \\ &= U(\mathcal{P}_n, f) - \frac{12}{n} \\ &= \left(2 + \frac{6}{n} \right) - \frac{12}{n}. \end{aligned}$$

Here we have used the result for $U(\mathcal{P}_n, f)$ found above; no use doing the same work twice! ■

Note 1 that in the Upper and Lower sums for an *arithmetic* partition the lengths of the sub-intervals do **not** depend on i, in fact $x_i - x_{i-1} = (b-a)/n$. So

$$U(\mathcal{P}_n, f) = \frac{b-a}{n} \sum_{i=1}^{n} M_i \quad \text{and} \quad L(\mathcal{P}_n, f) = \frac{b-a}{n} \sum_{i=1}^{n} m_i.$$

Note 2 If f is *increasing* on $[a,b]$ then

$$m_i = \operatorname*{glb}_{[x_{i-1},x_i]} f(x) = f(x_{i-1}) \quad \text{and} \quad M_i = \operatorname*{lub}_{[x_{i-1},x_i]} f(x) = f(x_i)\,.$$

Thus

$$m_i = f(x_{i-1}) = M_{i-1},$$

though only for $2 \le i \le m$. So with the arithmetic partition $\mathcal{P}_n$ of $[a,b]$,

$$\begin{aligned}
L(\mathcal{P}_n, f) &= \frac{b-a}{n}\sum_{i=1}^{n} m_i = \frac{b-a}{n}\left(m_1 + \sum_{i=2}^{n} m_i\right) \\
&= \frac{b-a}{n}\left(m_1 + \sum_{i=2}^{n} M_{i-1}\right) = \frac{b-a}{n}\left(m_1 + \sum_{i=1}^{n-1} M_j\right) \\
&= \frac{b-a}{n}\left(m_1 + \sum_{i=1}^{n} M_j - M_n\right).
\end{aligned}$$

Yet

$$\frac{b-a}{n}\sum_{i=1}^{n} M_j = U(\mathcal{P}_n, f)\,,$$

and, since f is increasing, $m_1 = f(a)$ and $M_n = f(b)$. Hence

$$L(\mathcal{P}_n, f) = U(\mathcal{P}_n, f) + \frac{b-a}{n}\left(f(a) - f(b)\right). \tag{1}$$

Be aware that we are assuming that f is increasing in which case $f(b) \ge f(a)$, i.e. $f(a) - f(b) \le 0$ so the above result fits in with the expectation that $L(\mathcal{P}_n, f) \le U(\mathcal{P}_n, f)$. This last result shows that we need only calculate one of $L(\mathcal{P}_n, f)$ or $U(\mathcal{P}_n, f)$, the other following quickly from (1). End of Note 2

Example 4.12 *Let*

$$f : [2,4] \to \mathbb{R} : x \mapsto \frac{1}{x^2}.$$

Find $U(\mathcal{Q}_n, f)$ and $L(\mathcal{Q}_n, f)$ for the geometric partitions $\mathcal{Q}_n$ of $[2,4]$.

Solution In the notation above $a = 2$, $b = 4$, so

$$\eta = \left(\frac{4}{2}\right)^{1/n} = 2^{1/n},$$

in which case $\eta^n = 2$. Then

$$\mathcal{Q}_n = \left\{2\eta^i : 0 \le i \le n\right\}.$$

A general sub-interval is

$$[x_{i-1}, x_i] = \left[2\eta^{i-1}, 2\eta^i\right]$$

and

$$x_i - x_{i-1} = 2\eta^i - 2\eta^{i-1} = 2\eta^i\left(1 - \frac{1}{\eta}\right).$$

It simplifies calculations if you write the difference in this way, with only **one** occurrence of the i.

In this example f is *decreasing* so

$$M_i = \operatorname*{lub}_{[x_{i-1},x_i]} f(x) = f(x_{i-1}) = \frac{1}{x_{i-1}^2} = \frac{1}{\left(2\eta^{i-1}\right)^2}.$$

Similarly,

$$m_i = \operatorname*{glb}_{[x_{i-1},x_i]} f(x) = f(x_i) = \frac{1}{x_i^2} = \frac{1}{\left(2\eta^i\right)^2}.$$

Since the expression for m_i is slightly simpler than that for M_i we start by examining the Lower Bound Sum

$$\begin{aligned}
L(\mathcal{Q}_n, f) &= \sum_{i=1}^{n} m_i\left(x_i - x_{i-1}\right) \\
&= \sum_{i=1}^{n} \frac{1}{\left(2\eta^i\right)^2} 2\eta^i\left(1 - \frac{1}{\eta}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{\eta}\right)\sum_{i=1}^{n}\frac{1}{\eta^i} \\
&= \frac{1}{2}\left(1 - \frac{1}{\eta}\right)\sum_{i=1}^{n}\left(\frac{1}{\eta}\right)^i = \frac{1}{2}\left(1 - \frac{1}{\eta}\right)\frac{\frac{1}{\eta}\left(1 - \left(\frac{1}{\eta}\right)^n\right)}{\left(1 - \frac{1}{\eta}\right)}
\end{aligned}$$

on summing the geometric series, using $\sum_{i=1}^{n} x^i = x\left(1 - x^n\right)/\left(1 - x\right)$. Continuing, using $\eta^n = 2$,

$$L(\mathcal{Q}_n, f) = \frac{1}{2\eta}\left(1 - \frac{1}{2}\right) = \frac{1}{4\eta}.$$

For the Upper Sum we start by noting that

$$M_i = \frac{1}{\left(2\eta^{i-1}\right)^2} = \frac{\eta^2}{\left(2\eta^i\right)^2} = \eta^2 m_i.$$

Then we can write the Upper Sum in terms of the Lower Sum as

$$\begin{aligned} U(\mathcal{Q}_n, f) &= \sum_{i=1}^{n} M_i\left(x_i - x_{i-1}\right) = \sum_{i=1}^{n} \eta^2 m_i\left(x_i - x_{i-1}\right) \\ &= \eta^2 L(\mathcal{Q}_n, f) = \frac{\eta}{4}. \end{aligned}$$

using the result for the lower sum; again no use doing the same work twice!
■

Returning to the Theory

Recall that for a partition $\mathcal{P}$ we defined

$$M_i = \text{lub}\left\{f(x) : x \in [x_{i-1}, x_i]\right\},$$

and

$$m_i = \text{glb}\left\{f(x) : x \in [x_{i-1}, x_i]\right\},$$

for $1 \leq i \leq n$.

Because we will use the following observation again we write it as a lemma:

Lemma 4.13 *If f is bounded on $I \subseteq \mathbb{R}$ and $J \subseteq I$ then*

$$\underset{I}{\text{glb}} f \leq \underset{J}{\text{glb}} f \quad \textit{and} \quad \underset{I}{\text{lub}} f \geq \underset{J}{\text{lub}} f.$$

Proof If $m = \text{glb}_I f$ then m is a lower bound for the set of values $f(x)$ for $x \in I$. Thus m is also *a* lower bound for f evaluated on $x \in J \subseteq I$. Hence m is less than or equal to the *greatest* of all lower bounds for the set of values of f as x varies over J. That is $m \leq \text{glb}_J f$. The first result now follows.

If $M = \text{lub}_I f$ then M is an upper bound for the set of values of $f(x)$ for $x \in I$. Thus M is *an* upper bound for f evaluated on $x \in J \subseteq I$. Hence M is greater than or equal to the *least* of all upper bounds for f on J. That is $M \geq \text{lub}_j f$. The second result follows. ■

With $I = [a, b]$ and $J = [x_{i-1}, x_i]$ we get $m \leq m_i$ and $M \geq M_i$ for all $i : 1 \leq i \leq n$. Combine as,

$$m \leq m_i \leq M_i \leq M$$

for $1 \leq i \leq n$.

Lemma 4.14 *For all partitions $\mathcal{P}$ of $[a,b]$ we have for any function f bounded on $[a,b]$,*

$$m\,(b-a) \le L(\mathcal{P},f) \le U(\mathcal{P},f) \le M\,(b-a)\,.$$

Proof Consider

$$m \le m_i \le M_i \le M$$

for all $1 \le i \le n$. Multiply through by $x_i - x_{i-1}$ and sum over $1 \le i \le n$ to get

$$m\sum_{i=1}^{n}(x_i - x_{i-1}) \le \sum_{i=1}^{n} m_i\,(x_i - x_{i-1}) \le \sum_{i=1}^{n} M_i\,(x_i - x_{i-1}) \le M\sum_{i=1}^{n}(x_i - x_{i-1})\,.$$

This is the required result since the sub-intervals, $[x_{i-1}, x_i]$ are **disjoint** (other than at their end points) and **cover** $[a,b]$ and so the sum of their lengths equals the length of $[a,b]$, i.e.

$$\sum_{i=1}^{n}(x_i - x_{i-1}) = b - a.$$

■

Recall that we are interested in the

- "smallest values" taken by $U(\mathcal{P},f)$ as $\mathcal{P}$ varies, and the
- "largest values" of $L(\mathcal{P},f)$ as $\mathcal{P}$ varies.

From Lemma 4.14 we see that, for f bounded on $[a,b]$, the set of real numbers

$$\{L(\mathcal{P},f) : \mathcal{P} \text{ a partition of } [a,b]\}$$

is a **non-empty set bounded above**, by $M(b-a)$. So by Completeness of $\mathbb{R}$ this set has a *least upper* bound.

Similarly, for f bounded on $[a,b]$, the set of real numbers

$$\{U(\mathcal{P},f) : \mathcal{P} \text{ a partition of } [a,b]\}$$

is a **non-empty set bounded below**, by $m(b-a)$. So by Completeness of $\mathbb{R}$ this set has a *greatest lower* bound.

Definition 4.15 *For f bounded on $[a, b]$, the **Upper Integral** is*

$$\overline{\int_a^b} f = \operatorname{glb}\{U(\mathcal{P}, f) : \mathcal{P} \text{ a partition of } [a, b]\} \tag{2}$$

*and the **Lower Integral** is*

$$\underline{\int_a^b} f = \operatorname{lub}\{L(\mathcal{P}, f) : \mathcal{P} \text{ a partition of } [a, b]\}.$$

Note that the upper and lower integrals can always be calculated for a bounded function on a bounded interval.

> **Advice for exam**. In remembering these definitions it is simple to recall that the *upper* integral is related to the *upper* sums and the *lower* integral to the *lower* sums. But students often get the glb and lub confused. You could think about the upper integral as the *least* (i.e. a lower bound) of all overestimates whilst the lower integral is the *largest* (i.e. an upper bound) of all underestimates.

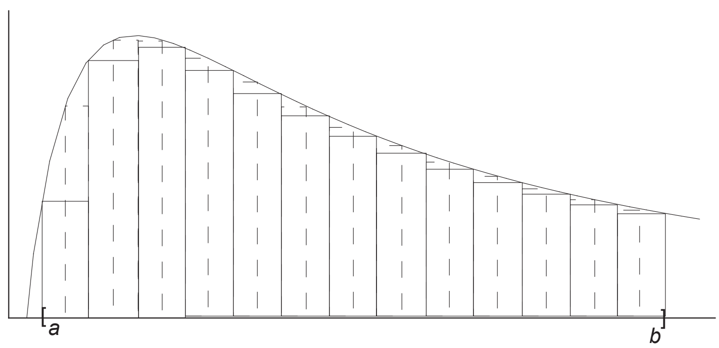
It is suggestive of the words used that the upper integral is greater than the lower integral, but this has to be proved.

Definition 4.16 *If $\mathcal{P}$ and $\mathcal{D}$ are two partitions of a set which satisfy $\mathcal{P} \subseteq \mathcal{D}$, we say that $\mathcal{D}$ is a **refinement** of $\mathcal{P}$. (We also say that $\mathcal{D}$ is **finer** than $\mathcal{P}$ and, equivalently, $\mathcal{P}$ is **coarser** than $\mathcal{D}$.)*

Proposition 4.17 *If f is bounded on $[a, b]$ and $\mathcal{D}$ is a refinement of $\mathcal{P}$, both partitions of $[a, b]$, then*

$$L(\mathcal{P}, f) \le L(\mathcal{D}, f) \quad \text{and} \quad U(\mathcal{D}, f) \le U(\mathcal{P}, f).$$

Diagram for lower sums. As we increase the number of points in the partition the lower sum increases.



Proof for lower bound sums. Let $\mathcal{P} = \{x_i : 0 \le i \le n\}$, so

$$L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i\,(x_i - x_{i-1})\,.$$

Choose any $y \in \mathcal{D} \setminus \mathcal{P}$. Thus there must exist $1 \le j \le n$ such that $x_{j-1} < y < x_j$. Then

$$L(\mathcal{P} \cup \{y\}\,, f) = \sum_{i \ne j} m_i\,(x_i - x_{i-1}) + \operatorname*{glb}_{[x_{j-1},y]} f\ (y - x_{j-1}) + \operatorname*{glb}_{[y,x_j]} f\,(x_j - y)\,. \quad (3)$$

We then use the facts that $[x_{j-1}, y]\,, [y, x_j] \subseteq [x_{j-1}, x_i]$ along with Lemma 4.13 to deduce

$$\operatorname*{glb}_{[x_{j-1},y]} f \ \ge\ \operatorname*{glb}_{[x_{j-1},x_j]} f\ = m_j \quad \text{and} \quad \operatorname*{glb}_{[y,x_j]} f \ \ge\ \operatorname*{glb}_{[x_{j-1},x_j]} f\ = m_j.$$

Hence RHS(3) is

$$\begin{aligned}
&\ge \sum_{i \ne j} m_i\,(x_i - x_{i-1}) + m_j\,(y - x_{j-1}) + m_j\,(x_j - y) \\
&= \sum_{i \ne j} m_i\,(x_i - x_{i-1}) + m_j\,\{(y - x_{j-1}) + (x_j - y)\} \\
&= \sum_{i \ne j} m_i\,(x_i - x_{i-1}) + m_j\,(x_j - x_{j-1}) = L\,(\mathcal{P}, f)\,.
\end{aligned}$$

That is, $L(\mathcal{P} \cup \{y\}\,, f) \ge L(\mathcal{P}, f)$. Continue adding in points from $\mathcal{D} \setminus \mathcal{P}$, to get $L(\mathcal{D}, f) \ge L(\mathcal{P}, f)$.

I leave the proof for upper bound sums, namely that $U(\mathcal{D}, f) \le U(\mathcal{P}, f)\,,$ to the students. ■

So, as you add in points to a partition the values of the lower sum increases, presumably getting closer to the value of the lower integral. Similarly the values of the upper sums decrease, hopefully getting closer to the value of the upper integral. Maybe by using a sequence of ever finer partitions one can say something about the upper and lower integrals.

Given any partition $\mathcal{P}$ it is trivially the case that $L(\mathcal{P}, f) \le U(\mathcal{P}, f)$. Yet far more is true.

Lemma 4.18 *For f bounded on $[a,b]$, and any two partitions $\mathcal{Q}$ and $\mathcal{R}$ of $[a,b]$ we have*

$$L(\mathcal{Q}, f) \leq U(\mathcal{R}, f)\,.$$

Proof Let $\mathcal{Q}$ and $\mathcal{R}$ be **any** two partitions of $[a,b]$. Then $\mathcal{Q} \cup \mathcal{R}$ is a refinement of both $\mathcal{Q}$ and $\mathcal{R}$. So

$$\begin{aligned} L(\mathcal{Q}, f) &\leq L(\mathcal{Q} \cup \mathcal{R}, f) \quad \text{by Lemma 4.17,} \\ &\leq U(\mathcal{Q} \cup \mathcal{R}, f) \quad \text{by Lemma 4.14} \\ &\leq U(\mathcal{R}, f) \quad \text{by Lemma 4.17 again.} \end{aligned}$$

■

As noted earlier, in using the words 'over' and 'under' you may be implicitly assuming the following result. It requires a proof!

Corollary 4.19 *If f is any bounded function on $[a,b]$ then*

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Proof Fix $\mathcal{R}$ and vary $\mathcal{Q}$. From $L(\mathcal{Q}, f) \leq U(\mathcal{R}, f)$ we thus see that $U(\mathcal{R}, f)$ is ***an*** upper bound for the set $\{L(\mathcal{Q}, f) : \mathcal{Q}\}$. But by definition $\underline{\int_a^b} f$ is the ***least*** of all upper bounds for this set, hence $\underline{\int_a^b} f \leq U(\mathcal{R}, f)$.

Now vary $\mathcal{R}$ and we see that $\underline{\int_a^b} f$ is ***a*** lower bound for the set $\{U(\mathcal{R}, f) : \mathcal{R}\}$. But by definition $\overline{\int_a^b} f$ is the ***greatest*** of all lower bounds on this set, so

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

as required. ■

Definition 4.20 *A bounded function f on $[a,b]$ is **Riemann integrable over** $[a,b]$ if*

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

The common value is called the (Riemann) integral and is denoted by $\int_a^b f$ or $\int_a^b f(x)\,dx$.

Note To save time in lectures I will often write R-$\int$ation and R-$\int$able in place of integration and integrable.

Basic Result For the examples below we need that, for any partition $\mathcal{P}$ of $[a,b]$, we have

$$\begin{aligned} L(\mathcal{P},f) &\le \underline{\int_a^b} f \qquad \text{by definition of lower integral,} \\ &\le \overline{\int_a^b} f \qquad \text{by the Corollary} \\ &\le U(\mathcal{P},f) \qquad \text{by definition of upper integral.} \end{aligned}$$

Thus we get the fundamental

$$L(\mathcal{P},f) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(\mathcal{P},f)\,. \tag{4}$$

Advice for exams. Remember this!

Return to Examples

Example 4.21 *Let*

$$f:[2,4] \to \mathbb{R} : x \mapsto 3x-8.$$

Prove, by verifying the definition, that f is Riemann integrable over $[2,4]$. What is the value of the integral?

Solution We have already calculated the Upper and Lower sums with $\mathcal{P} = \mathcal{P}_n$, the arithmetic partitions. Substituting those earlier results into (4) gives

$$2 - \frac{6}{n} \le \underline{\int_2^4} f(x)\,dx \le \overline{\int_2^4} f(x)\,dx \le 2 + \frac{6}{n}$$

for all $n \ge 1$. Let $n \to \infty$ to see that

$$2 \le \underline{\int_2^4} f(x)\,dx \le \overline{\int_2^4} f(x)\,dx \le 2,$$

which can only be true if we have equality throughout. In particular we have equality in the centre, i.e. $\underline{\int_2^4} f(x)\,dx = \overline{\int_2^4} f(x)\,dx$, and so, by definition, the function is Riemann integrable over $[2,4]$. The common value, 2, is therefore the value of the integral. ■

Advice for exam. To prove a function is integrable and then find the value of the integral you must not only remember (4) but also the end of the proof, namely let $n \to \infty$ to get a common value for the lower and upper integrals and tell me that this (1) verifies the definition that the function is integrable and (2) that this common value is the value of the integral.

Example 4.22 *Let*

$$f : [2, 4] \to \mathbb{R} : x \mapsto \frac{1}{x^2}.$$

Prove, by verifying the definition, that f is Riemann integrable over $[2, 4]$. What is the value of the integral?

Solution We have worked out the Upper and Lower sums with geometric partitions $\mathcal{Q}_n$. Substituting those earlier results into (4) gives

$$\frac{1}{4\eta} \leq \underline{\int_2^4} \frac{dx}{x^2} \leq \overline{\int_2^4} \frac{dx}{x^2} \leq \frac{\eta}{4},$$

for all $n \geq 1$, where $\eta = 2^{1/n}$. Let $n \to \infty$ when $\eta \to 1$ to see that we must have equality in the centre, i.e.

$$\underline{\int_2^4} \frac{dx}{x^2} = \overline{\int_2^4} \frac{dx}{x^2},$$

and so, by definition, the function is Riemann integrable over $[2, 4]$. The common value, $1/4$, is therefore the value of the integral. ■

Important The upper and lower integrals exist for all functions but not all functions are Riemann integrable.

Example 4.23 *Let $f : [0, 1] \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational.} \end{cases}$$

*Show that f is **not** Riemann integrable over $[0, 1]$.*

Solution Let $\mathcal{P}$ be any partition of $[0,1]$. Then in any sub-interval $[x_{i-1}, x_i]$ we can find an irrational number, so $m_i = 0$, and find a rational number, so $M_i = 1$. Thus

$$L(\mathcal{P}, f) = \sum_{i=1}^{n} 0\,(x_i - x_{i-1}) = 0 \quad \text{and} \quad U(\mathcal{P}, f) = \sum_{i=1}^{n} 1\,(x_i - x_{i-1}) = 1,$$

the last result following from the sum of the lengths of the sub-intervals equals the length of the total interval $[0,1]$.

Hence

$$\underline{\int_0^1} f(x)\,dx = \text{lub}\,\{L(\mathcal{P}, f) : \mathcal{P}\} = \text{lub}\,\{0\} = 0.$$

Similarly,

$$\overline{\int_0^1} f(x)\,dx = \text{glb}\,\{U(\mathcal{P}, f) : \mathcal{P}\} = \text{glb}\,\{1\} = 1.$$

Since the lower and upper integrals are different we deduce that f is not Riemann integrable over $[0,1]$. ■

This result is not ideal. This function differs from the function that is zero throughout $[0,1]$ on only a countable number of points. Since the zero function is integrable, it would be nice if $f(x)$ were integrable.

Return to Theory

If there were more time I would prove the following results.

Theorem 4.24 *If f is continuous on $[a,b]$ then f is Riemann integrable on $[a,b]$.*

Proof Not given. ■

Theorem 4.25 *If f is monotonic on $[a,b]$ then f is Riemann integrable on $[a,b]$.*

Proof Not given but should not be hard for the interested student for it follows quickly from the

$$L(\mathcal{P}_n, f) = U(\mathcal{P}_n, f) + \frac{b-a}{n}\,(f(a) - f(b))$$

seen earlier. ■

Theorem 4.26 *If the bounded function f is Riemann integrable on $[a,b]$ then $|f|$, defined as $|f|(x) = |f(x)|$, is also Riemann integrable over $[a,b]$ and*

$$\left|\int_a^b f\right| \le \int_a^b |f|. \tag{5}$$

Proof Not given. ■

The final result of the course concerns the relationship between integration and differentiation.

Theorem 4.27 ***Fundamental Theorem of Calculus****.*
Assume f is bounded on $[a,b]$.

1) If f is Riemann Integrable on $[a,b]$ then

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$.

2) Further, if f is continuous on $[a,b]$ then F is differentiable on (a,b) and $F'(x) = f(x)$ for all $x \in (a,b)$.

Proof
1) We wish to verify the $\varepsilon - \delta$ definition that F is continuous on $[a,b]$. We do it here only for the interior points, i.e. those in (a,b). I leave it to the interested student to check the definition at the end-points a and b using one-sided limits.

We are assuming that f is bounded thus there exists $N > 0 : |f(x)| \le N$ for all $x \in [a,b]$.

Let $c \in (a,b)$ be given. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/N > 0$. Assume $|x - c| < \delta$.

Split into two cases. If $x > c$ consider

$$\begin{aligned} |F(x) - F(c)| &= \left|\int_a^x f(t)\,dt - \int_a^c f(t)\,dt\right| \\ &= \left|\int_c^x f(t)\,dt\right| \\ &\le \int_c^x |f(t)|\,dt \qquad \text{by (5)} \\ &\le \int_c^x N dt = N(x-c). \end{aligned}$$

If $x < c$ consider $|F(x) - F(c)| = |F(c) - F(x)| \leq N(c - x)$ by above with c and x interchanged (since $c > x$). So, in all cases,

$$|F(x) - F(c)| \leq N|x - c| < N\delta = \varepsilon.$$

Hence we have verified the definition that F is continuous at c and thus on (a, b).

2) We wish to verify the $\varepsilon - \delta$ definition that

$$\lim_{x \to c} \frac{F(x) - F(c)}{x - c} = f(c)$$

for all points c in (a, b)

Let $c \in (a, b)$ be given. Let $\varepsilon > 0$ be given.

We are told that f is continuous at c in which case there exists $\delta > 0$ such that if $|t - c| < \delta$ then $|f(t) - f(c)| < \varepsilon$.

Assume x satisfies $0 < |x - c| < \delta$.

Split into two cases. If $x > c$ consider

$$\begin{aligned}
\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| &= \frac{1}{|x - c|} |(F(x) - F(c)) - (x - c) f(c)| \\
&= \frac{1}{|x - c|} \left| \int_c^x f(t)\, dt - \int_c^x f(c)\, dt \right| \\
&\leq \frac{1}{|x - c|} \int_c^x |f(t) - f(c)|\, dt \qquad \text{by (5)} \\
&\leq \frac{1}{|x - c|} \int_c^x \varepsilon dt = \varepsilon.
\end{aligned}$$

Here we have used the fact that $c \leq t \leq x$ implies $|t - c| \leq |x - c| < \delta$ which in turn implies $|f(t) - f(c)| < \varepsilon$, used inside the integral.

If $x < c$ the same result follows on writing

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| = \left| \frac{F(c) - F(x)}{c - x} - f(c) \right|.$$

So in all cases

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| < \varepsilon.$$

Thus

$$\lim_{x \to c} \frac{F(x) - F(c)}{x - c}$$

exists with limit $f(c)$, i.e. F is differentiable at c, with derivative $f(c)$. ■

That f is integrable is not sufficient to say that $F(x) = \int_a^x f(t)\,dt$ is differentiable let alone that it satisfies $F'(x) = f(x)$. We can see this by an example.

Example 4.28 *Define f on $[0, 2]$ by*

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 < x \le 2. \end{cases}$$

What is $F(x)$? What is $F'(1)$?

Solution left to student. ■

In this example f is not continuous at $x = 1$. Thus, we need the extra condition that f is continuous which gives us both that $F(x)$ is differentiable with $F'(x) = f(x)$.

How to use the Fundamental Theorem of Calculus? The second part of the theorem gives a way of evaluating $\int_a^x f(t)\,dt$ for continuous f.

Example Consider

$$f(x) = \frac{1}{\sqrt{x^2 + 1}},$$

which is continuous on $\mathbb{R}$. Therefore $F(x) = \int_0^x f(t)\,dt$ satisfies $F'(x) = f(x)$ for all $x \in \mathbb{R}$. Yet, on Problem Sheet 3 we found that

$$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}$$

for all $x \in \mathbb{R}$. So

$$\frac{d}{dx} F(x) = \frac{d}{dx} \sinh^{-1} x, \text{ i.e. } \frac{d}{dx} \left(F(x) - \sinh^{-1} x \right) = 0$$

for all $x \in \mathbb{R}$. The only function whose derivative is zero for all x is the constant function. Hence

$$F(x) - \sinh^{-1} x = c$$

for some $c \in \mathbb{R}$. That is

$$\int_0^x f(t)\,dt = F(x) = \sinh^{-1} x + c.$$

What is the value of c? Put $x = 0$ to find $c = \sinh^{-1} 0 - F(0) = 0$. Thus

$$\int_0^x \frac{dt}{\sqrt{t^2+1}} = \sinh^{-1} x.$$

(Of course, you could evaluate this integral by substitution, but we haven't had time in this course to justify substitution.) ■

Definition 4.29 *If f is continuous on (a,b) and F is continuous on $[a,b]$ and differentiable on (a,b) with $F'(x) = f(x)$ for all $x \in (a,b)$ then F is a* ***primitive for*** *f.*

Note that we say that F is **a** primitive for f, not **the** primitive. This is because constants vanish under differentiation and so if F is a primitive for f then so is $F(x)+c$ for any constant c. But more is true, if F_1 and F_2 are two primitives for f then $F_1'(x) = f(x) = F_2'(x)$ and so $(F_1 - F_2)'(x) = 0$. Thus $F_1(x) - F_2(x) = c$ for some constant c. Hence if F is a primitive then so is $F + c$ for any constant c and all primitives are of this form.

Hence, if you have a function continuous on $[a,b]$ you look in the "big book" of derivatives to find a derivative equal to your function. You then know that the integral of your function is equal (up to a constant) to the function of which you took the derivative to get $f(x)$. Further examples of this nature can be found on the problem sheet.